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# BANK LEVERAGE, FINANCIAL FRAGILITY AND PRUDENTIAL REGULATION

Olivier Bruno<sup>\*</sup>, André Cartapanis<sup>#</sup>, Eric Nasica<sup>+</sup>

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**Abstract:** We analyse the determinants of banks' balance-sheet and leverage-ratio dynamics and their role in increasing financial fragility. Our results are twofold. First, we show that there is a value of bank's leverage that minimises financial fragility. Second, we show that this value depends on the overall business climate, the expected value of the collateral and the riskless interest rate. This result leads us to advocate the establishment of an adjustable leverage ratio, depending on economic conditions, rather than the fixed ratio provided for under the new Basel III regulation.

**Keywords:** Bank Leverage, Leverage ratios, Financial Instability, Prudential Regulation.

**JEL Code:** E44, G28

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<sup>\*</sup> University of Nice Sophia-Antipolis, GREDEG-CNRS, SKEMA Business School and OFCE-DRIC.  
Corresponding author: 250, rue Albert Einstein, Valbonne Sophia-Antipolis, 06 560, France.  
e-mail: bruno@gredeg.cnrs.fr, Tel: 00.33.4.93.95.43.81., fax: 00.33.4.93.65.37.98.

<sup>#</sup> Sciences Po Aix-en-Provence, GREDEG-CNRS and CHERPA-Sciences Po Aix-en-Provence, e-mail: andre.cartapanis@sciencespo-aix.fr

<sup>+</sup> University of Nice Sophia-Antipolis, GREDEG-CNRS, e-mail: nasica@gredeg.cnrs.fr

## 1. INTRODUCTION

The devastating consequences of the 2008 financial crisis on economic activity and unemployment have reignited the debate on the causes of financial fragility and instability. Empirically, a thorough overview of the events preceding and accompanying the current financial crisis is provided in Allen and Carletti (2010), Brunnermeier (2009), Greenlaw et al. (2008) and Taylor (2009). The financial crisis is attributed to a number of factors associated with the housing and credit markets. Suggested causes include the inability of homeowners to make their mortgage payments, overbuilding during the boom period, high personal and corporate debt levels, financial product innovation, the failure of key financial institutions and errors of judgment by credit rating agencies in the rating of structured products. Macroeconomic factors such as accommodating monetary policy, global imbalances and government regulation (or lack thereof) are also considered to have played a direct or indirect role in the crisis (Cabral, 2013).

Another important factor underlined is the significant increase in banks' leverage levels in the 4-5 years preceding the crisis of the summer of 2007 and the panic of the autumn of 2008, in particular for the major European banks and for US investment banks. The increase in bank leverage was around 50% in some cases. The levels of asset-to-equity ratios (or *equity multiplier*) remained quite close to a range of 20-25, i.e. capital-asset ratios (or *leverage ratios*) of 5% to 4%, until 2003-2004, with significant differences according to regions or categories of banks.<sup>1</sup> Between 2005 and the crisis, the top 50 major global banks, of the US investment banks and European universal banks, had an equity multiplier close to or even exceeding 30, and therefore a leverage ratio of only 3% (Financial Stability Forum and Committee on the Global Financial System Joint Working Group (2009)).

This excess leverage prior to the crisis and the devastating impact of deleveraging in its wake explain why the G20 and all the supervisors were converted to the idea that a leverage ratio should be added to the traditional prudential measures. This would be complementary to prudential risk ratios, and consequently not replace Basel II or Basel III micro-prudential regulation under preparation. This leverage ratio is a measure of a bank's Tier 1 capital as a percentage of its assets plus off-balance sheet exposures and derivatives. The Basel Committee has chosen a minimum leverage ratio of 3%, and thus a maximum equity multiplier of 33. The implementation of this ratio on an experimental basis is set to begin in January 2013, and after various adjustment phases between 2015 and 2017, this ratio will become imperative, in Pillar I of Basel III, in January 2018 (BIS Annual Report, 2011).

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<sup>1</sup> In this paper, we will always use the term "equity multiplier" to characterise the "asset-to-equity ratio" and the term "leverage ratio" to refer to the "capital-asset ratio".

However, the question of the efficiency of such a regulatory leverage ratio is open to debate. If the chosen value of the ratio is too low, it will have a detrimental impact on banks' ability to make loans. On the contrary, if the chosen value of the ratio is too high, it will not prevent banks' excess risk taking. In both cases, the question of an existing value of leverage ratio that minimizes banks' financial fragility<sup>2</sup> and its link with credit availability must be addressed. This is the main objective of this paper.

Our contribution is related to the literature on bank leverage and financial fragility. A major author dealing with this question is Minsky (1982, 1986) who developed a business cycle theory based on a financial conception of economic fluctuations and more specifically the “financial instability hypothesis”. In Minsky’s approach, banks' profit-seeking behaviour leads them to deliberately reduce their capital-asset ratio and engage in financial operations involving high leverage when their activity is expanding. As underlined by contemporary economists such as Goodhart (2010) or Roubini and Mihm (2010), the recent financial crisis is largely based on similar mechanisms. More recently, several economists have assigned great importance to debt leverage in the dynamics of financial instability. J. Geanakoplos (2010a, 2010b) postulates a leverage cycle, seen as a recurrent phenomenon in US financial history. In a series of articles on the subprime crisis, T. Adrian and H. S. Shin (2010a, 2010b) examine the role of financial intermediation in the financial crisis of 2007-2009, and at the same time the role of leverage effects. They emphasize the pro-cyclicality of leverage and the positive relationship between leverage and the size of financial intermediaries’ balance sheets, especially before the crisis. In the same vein, H.S. Shin (2009) models a lending boom fuelled by declines in measured risk. He shows that in benign financial market conditions when measured risks are low, financial intermediaries expand their balance sheets as they increase leverage. There is, of course, a symmetrical process that accentuates the magnitude of the crisis when the measured risks are high, leading to sharp deleveraging, then resulting in a credit crunch.

Following this growing literature, we develop a model that shows that financial fragility can result from banks’ optimal profit maximisation decisions even if they meet the micro-prudential requirements laid down by the Basel II accord. However, in addition to the previous literature, our model allows us to address the question of an “optimal” leverage ratio that minimises financial fragility.

The question for the need of such a leverage ratio restriction was already studied by the literature. In a seminal paper, Blum (2008) shows that in a Basel II framework, banks can report their level of risk untruthfully. In this context, a risk-independent leverage ratio restriction may be necessary to induce truthful risk reporting. However, Blum does not propose to evaluate the value of such a ratio. More recently, Jarrow (2013) tries to

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<sup>2</sup> We define bank fragility as the critical level of macroeconomic shock at which a bank goes bankrupt.

provide a rationale for determining the value of a maximal leverage ratio based on a VaR rules. In his paper, this value depends on banks' microeconomics characteristics and especially on the structure of banks' balance sheet. Our approach differs from the one of Jarrow (2013) since in our model, the optimal value of a leverage ratio depends also on the macroeconomics condition and is not only linked with the specific characteristics of one bank. Two main results are obtained from our analysis.

First, we show that there is a non-linear relationship between the level of bank leverage and financial fragility defined as the critical level of macroeconomic shock at which a bank goes bankrupt. More precisely, there is an optimal value of leverage minimising financial fragility which allows us to identify two states called respectively the "inefficient equilibrium state" and the "trade-off equilibrium state". In the inefficient equilibrium state, high levels of financial fragility are associated with low bank leverage and low levels of credit availability. On the contrary, in the trade-off equilibrium state, high levels of financial fragility are associated with high bank leverage and high levels of credit availability. This result allows us to stress that bank leverage may increase without being detrimental to financial instability as long as the level of leverage chosen by the banks is lower than the one that minimises financial fragility. This result is useful also in order to understand the potential impact of the new Basel III capital regulation that introduces a maximum value for bank leverage. Actually, if the maximum value fixed by the regulatory authorities is too low, the economy can be trapped in the inefficient equilibrium state, whereas excessively high maximum leverage stimulates credit availability to the detriment of financial stability.

Second, we show that the equilibrium level of leverage chosen by the bank and the value of leverage that minimises financial fragility both depend on the overall economic situation and on the value of the collateral provided by firms to the bank. We accordingly show that there is a critical threshold above which an increase in the expected value of collateral leads to an increase in financial fragility. We link such a result to the increase in financial fragility due to a rise in assets prices. This result is in accordance with the literature that explicitly links bank behaviour, endogenous debt growth and financial instability. Compared to this literature, the originality of our analysis is to show that even in an "ideal" economic environment (perfect information, economic expansion, optimistic expectations, rising assets prices, rational agents within the standard meaning of the term), a pro-cyclical financial fragility process based on the relationship between assets prices and the bank lending cycle may develop. As a result, since the optimal level of leverage minimising financial fragility depends on the overall business climate, we advocate the establishment of an adjustable leverage ratio, depending on economic conditions, rather than the fixed ratio provided for under Basel III.

The rest of the paper is organised as follows. Section 2 presents the model, while Sections 3 and 4 show and discuss our main results. Section 5 concludes.

## 2. THE MODEL

We consider three classes of agents – firms, individual investors and a bank – and two periods. In the first period, firms need external funds in order to invest in a risky project subject to a macroeconomic shock. We assume that firms have access to bank loans only. In this period, financial contracts are signed between lenders and the bank and investment decisions are made. In the second period, the value of the macroeconomic shock and the effective return on investment are known. Non-defaulting firms have to pay for their external funds whereas defaulting firms are liquidated. We assume that all parties are risk-neutral and protected by limited liability.

### 2.1. Firms and bank behaviours

Firms have access to a risky investment project that requires one unit of wealth in order to be undertaken in period 1. There is no financial market and firms lack capital and need to borrow the total amount of their investment from a bank.

In period 2, the total return on investment projects  $(V_i)$  undertaken by firms  $i$  (we assimilate a firm with its project) depends on two parameters. The first one is a specific factor  $x_i$ , related to firms' specific characteristics. We assume that  $x_i$  is uniformly distributed over  $[0,1]$  and known at the beginning of period 1 by firms and by the bank. The second is a macroeconomic shock  $\theta$  which is similar to systematic risk for projects. Thus, the total return on project  $i$  at period 2 is given by  $V_i = \theta x_i$ .

We adopt  $\theta = \bar{\theta} + dz$  with  $\bar{\theta} > 1$  and  $dz = \varepsilon \sigma_\theta \sqrt{dt}$  a Brownian motion.  $\sigma_\theta$  is a measure of the volatility of the shock,  $\varepsilon \sim N(0,1)$  is a normally distributed stochastic variable and  $dt = 1$  (the length of the period). Consequently, the value of the shock in period 2 is equal to  $\theta = \bar{\theta} + \varepsilon \sigma_\theta$  and depends only on the realisation of the stochastic variable  $\varepsilon$ . As  $E[\varepsilon] = 0$ , the expected value of the macroeconomic shock in period 1 is given by  $E[\theta] = \bar{\theta}$ .

Lastly, we assume that firms must provide an asset as collateral for their loan, with  $Z^e$  being the expected value of this collateral at period 1.  $Z^e$  is supposed to be the same for all firms with  $Z^e \in ]0,1[$ .

There is one bank in the economy endowed with its own capital and individual investors' deposits. These deposits are insured through a government-funded scheme and receive the risk-free return  $\gamma > 1$  (which is also the opportunity cost of the funds for the bank). The bank is owned by shareholders who provide it with the equity capital

required by banking regulations. At the beginning of period 1, the bank decides to finance the various investment projects proposed by firms at a return  $R > \gamma$ . There is no moral hazard between the bank and firms as the value of  $x_i$  is common knowledge at period 1 and the realisation of  $V_i$  is freely observable by the bank at period 2.

Finally, we assume that all firms apply for credit in period 1 and that the bank finances them as long as the expected value of their projects exceeds the rate of return they must pay back in period 2, such that

$$E[V_i] = \bar{\theta} x_i \geq R \quad (1)$$

From equation (1) it is easy to show that the last firm financed by the bank is given by

$$\bar{x}_i(R) = \frac{R}{\bar{\theta}} \quad (2)$$

and the total quantity of financing in the economy is given by

$$D = (1 - \bar{x}_i(R)) \text{ with } \bar{\theta} > R \quad (3)$$

## 2.2. Bank regulatory capital and firms' probability of default

We assume that the bank's level of capital is exactly equal to that required by the Basel II Advanced IRB approach.<sup>3</sup> According to Basel II, the level of a bank's capital requirements is linked to the level of risk of its loan portfolio. In order to calculate the bank's capital requirements, we need to calculate the probability of default of each project (or equivalently firm) it finances.

A project  $i$  is in default if the firm cannot repay the value  $R$  in period 2. Formally, the probability of default of each project is given by the following conditional probability

$$P[V_i < R \mid x_i \geq \bar{x}_i(R)] \quad (4)$$

As  $V_i = \theta x_i = \bar{\theta} x_i + x_i \varepsilon \sigma_\theta$  we obtain

$$\begin{aligned} & P[\bar{\theta} x_i + x_i \varepsilon \sigma_\theta < R \mid x_i \geq \bar{x}_i(R)] \text{ and} \\ & P\left[\varepsilon < \bar{\varepsilon}_i = \frac{R - \bar{\theta} x_i}{x_i \sigma_\theta} \mid x_i \geq \bar{x}_i(R)\right] = \bar{p}_i = \phi(\bar{\varepsilon}_i(R)) \end{aligned} \quad (5)$$

where  $\phi$  denotes the cumulative distribution function of a standard normal random variable. This equation means that project  $i$  defaults if the realised value of the shock  $\varepsilon$

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<sup>3</sup> This assumption does not change the main results of the paper and can be justified by the fact that capital is more costly than deposits.

is larger than the critical value  $\bar{\varepsilon}_i$ , with  $\bar{\varepsilon}_i \leq 0$  since  $R \leq \bar{\theta}x_i$ . This probability of default is an increasing function of the rate of return ( $R$ ) charged by the bank, and of the volatility of the shock,  $\sigma_\theta$ . Conversely, this probability of default decreases in line with the value of  $x_i$  i.e. the intrinsic “quality” of project  $i$ .

According to the Basel II Advanced IRB approach, the level of a bank’s capital requirements depends on the level of risk of its loan portfolio. Using the usual single-factor risk model adopted by the IRB approach allows us to define the conditional credit default rate  $P_i(\alpha, \bar{\varepsilon}_i)$  as

$$P_i(\alpha, \bar{\varepsilon}_i) = \phi \left[ \frac{\phi^{-1}(\bar{p}_i) + \sqrt{\rho} \phi^{-1}(\alpha)}{\sqrt{(1-\rho)}} \right] \quad (6)$$

According to equation (6), there is an  $\alpha$  chance that the default rate on the loan  $i$  will be less than  $P_i(\alpha, \bar{\varepsilon}_i)$ . We use a measure of  $\rho$  and  $\alpha$  similar to that adopted by the IRB

approach with  $\alpha = 0.999$  and  $\rho = 0.24 - 0.12 \left[ \frac{1 - \exp(-50\bar{p}_i)}{1 - \exp(-50)} \right]$ .

$\phi^{-1}(\bar{p}_i) = \bar{\varepsilon}_i$ ,  $\phi^{-1}(\alpha) = \varepsilon_\alpha$  and equation (6) becomes

$$P_i(\alpha, \bar{\varepsilon}_i) = \phi \left[ \frac{\bar{\varepsilon}_i + \sqrt{\rho} \varepsilon_\alpha}{\sqrt{(1-\rho)}} \right] \quad (7)$$

In order to compute the regulatory capital requirements, we need the Exposure at Default (EAD) and the Loss Given Default (LGD) of the bank. In our model, EAD is equal to 1 (the amount of the loan) while we assume that the LGD is given by  $\lambda(Z^e)$

with  $\frac{\partial \lambda(Z^e)}{\partial Z^e} < 0$  and  $\lim_{Z^e \rightarrow 1} \lambda(Z^e) = 0$ . These assumptions are in accordance with the

Advanced IRB approach where the LGD is chosen by the bank according to its internal model and the estimated value of the debt recovery rate (positively correlated with the expected value of the collateral). Consequently, Basel II bank capital requirements for project  $i$  are given by the following formula:<sup>4</sup>

$$k_i(R) = \lambda(Z^e)(P_i(\alpha, \bar{\varepsilon}_i) - \bar{p}_i) \quad (8)$$

Consequently, the total amount of the bank’s capital (the value of its Economic Capital) is equal to the regulatory capital required in order to cover its loan portfolio:

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<sup>4</sup> We do not take maturity adjustments into consideration.



$$\begin{aligned}
K(R) &= \int_{\bar{x}_i(R)}^1 k_i(R) dx_i = \int_{\bar{x}_i(R)}^1 \lambda(Z^e) (P_i(\alpha, \bar{\varepsilon}_i) - \bar{p}_i) dx_i \\
K(R) &= \int_{\bar{x}_i(R)}^1 \lambda(Z^e) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i
\end{aligned} \tag{9}$$

### 3. EQUILIBRIUM AND FINANCIAL STABILITY

We have defined the equilibrium of the economy as the value of the rate of return that maximises the bank's Risk Adjusted Return on Capital (RAROC). This equilibrium value of the rate of return is associated with a certain bank leverage level and a given level of financing.

#### 3.1. Bank's equilibrium and the total value of financing

The expected profit of the bank depends on the number of projects it finances and the number of defaulting loans. When the bank finances a project, it expects to receive the rate of return  $R$  if the project succeeds and the expected value of the collateral  $Z^e$  if the project fails.<sup>5</sup> Project  $i$ 's probability of default is given by  $\phi(\bar{\varepsilon}_i)$  and its probability of success is given by  $1 - \phi(\bar{\varepsilon}_i)$ . Moreover, we know that the last project financed by the bank is  $\bar{x}_i(R)$  and that the cost of deposit for the bank (opportunity cost of the funds) is given by  $\gamma > 1$ . The expected profit of the bank at period 1, net of the opportunity cost of the funds, is equal to

$$\begin{aligned}
\Pi_b^e(R) &= \int_{\bar{x}_i(R)}^1 [(1 - \phi(\bar{\varepsilon}_i))R + \phi(\bar{\varepsilon}_i)Z^e] dx_i - \int_{\bar{x}_i(R)}^1 [\gamma(1 - k_i(R))] dx_i \\
\Pi_b^e(R) &= \int_{\bar{x}_i(R)}^1 [(1 - \phi(\bar{\varepsilon}_i))R + \phi(\bar{\varepsilon}_i)Z^e - \gamma] dx_i + \gamma \int_{\bar{x}_i(R)}^1 k_i(R) dx_i \text{ or using (9)} \\
\Pi_b^e(R) &= \int_{\bar{x}_i(R)}^1 [(1 - \phi(\bar{\varepsilon}_i))R + \phi(\bar{\varepsilon}_i)Z^e - \gamma] dx_i + \gamma K(R)
\end{aligned} \tag{10}$$

We assume that the bank seeks to maximise its net expected RAROC. The gross expected RAROC is defined by

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<sup>5</sup> We assume the project has no residual value in the event of default.

$$\text{RAROC} = R_b = \frac{\text{Expected profit of the bank}}{\text{Economic Capital}}$$

Using (9) and (10), the value of the gross expected RAROC is equal to

$$R_b = \frac{\Pi_b^e(R)}{K(R)} = \frac{\int_{\bar{x}_i(R)}^1 \left[ (1 - \phi(\bar{\varepsilon}_i)) R + \phi(\bar{\varepsilon}_i) Z^e - \gamma \right] dx_i}{\int_{\bar{x}_i(R)}^1 \lambda(Z^e) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i} + \gamma \quad (11)$$

and the net expected RAROC is equal to

$$\Gamma(R) \equiv (R_b - \gamma) = \frac{\int_{\bar{x}_i(R)}^1 \left[ (1 - \phi(\bar{\varepsilon}_i)) R + \phi(\bar{\varepsilon}_i) Z^e - \gamma \right] dx_i}{\int_{\bar{x}_i(R)}^1 \lambda(Z^e) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i} \quad (12)$$

PROPOSITION 1.

a. For  $2\gamma - Z^e < \bar{\theta} < \frac{5\gamma\sigma_\theta}{(\gamma - Z^e)}$ , there is a unique value  $R^* \in ]R_c, \bar{\theta}[$  with  $R_c > \gamma$  that maximises the net RAROC of the bank and  $\Gamma(R^*) > 0$ .

b. The total level of financing in the economy is given by  $D^*(R^*) = (1 - \bar{x}_i(R^*)) > 0$

with  $\frac{\partial D(R)}{\partial R} < 0, \forall R$  and the equilibrium level of the bank's leverage (or equity

multiplier) is equal to  $\ell^*(R^*) = \frac{D^*(R^*)}{K^*(R^*)}$  with  $\frac{\partial \ell(R)}{\partial R} < 0, \forall R$ .

Proof of Proposition 1: see Appendix.

According to part a of proposition 1, the bank can always find a unique value of the rate of return it charges to firms that maximises its net expected RAROC. Moreover, as stated in part b of proposition 1, the total quantity of financing in the economy is a decreasing function of the rate of return charged by the bank, since when the bank's rate of return on loans falls, some new firms find it profitable to apply for credit. Finally, the level of bank leverage is a decreasing function of the rate of return it charges to firms.

This result can be easily understood as firms are financed partly by bank capital and partly by deposits. Thus, when the bank cuts the rate of return it charges to firms, its level of assets increases (since the amount of loans financed increases) at a faster pace than its level of regulatory capital, leading to a rise in its equilibrium level of leverage.

### 3.2. Bank leverage and financial fragility

We now consider the value of the macroeconomic shock that leads the bank to default. We assume that the bank goes bankrupt when its value at period 2 is lower than the level of capital required by the regulations. In that case, the bank can be shut down by the regulatory authorities. This view is similar to the one proposed by Heid (2007).<sup>6</sup>

The value of the bank at period 2 comprises two parts. The first part is the capital endowment that allows the bank to absorb part of the macroeconomic shock, while the second part is determined by the value of the bank's assets. Bank capital consists of regulatory capital only. The value of the bank's assets is linked to the realised value of the macroeconomic shock and the realised value of the collateral  $Z$ . We assume, in line with empirical literature on this question, that the realised value of the collateral is constant at a level  $\bar{Z}$  in line with the historical debt recovery rate on investment projects.<sup>7</sup> Thus, the value of the bank at period 2 essentially depends on the realisation of the macroeconomic shock.

$\varepsilon_c < 0$  is defined as the value of the macroeconomic shock for which firms with  $\bar{x}_i < x_i \leq x_c = \frac{R}{\theta + \sigma_\theta \varepsilon_c}$  are in default. This means that financed firms with  $x_i \in ]x_c, 1]$  are successful whereas financed firms with  $x_i \in [\bar{x}_i, x_c]$  are in default. The equilibrium value of the bank at period 2 is given by

$$V_b(R) = K(R) + \int_{x_c(R)}^1 R dx_i + \int_{\bar{x}_i(R)}^{x_c(R)} \bar{Z} dx_i - \int_{\bar{x}_i(R)}^1 \gamma dx_i \quad (13)$$

Or equivalently with  $K(R) = \int_{\bar{x}_i(R)}^1 \lambda(\bar{Z}) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i$ ,

$$x_c(R) = \frac{R}{\theta + \varepsilon_c \sigma_\theta}, \quad \bar{x}_i(R) = \frac{R}{\theta}$$

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<sup>6</sup> Heid (2007) pp. 3888-3889.

<sup>7</sup> Moody's or S&P's evaluations of the mean debt recovery rate, for instance.

$$V_b(R) = \int_{\bar{x}_i(R)}^1 \lambda(\bar{Z}) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i + R[1 - x_c(R)] + \bar{Z}[x_c(R) - \bar{x}_i(R)] - \gamma[1 - \bar{x}_i(R)]$$

Consequently, the bank goes bankrupt when

$$V_b(R) = K(R) = \int_{\bar{x}_i(R)}^1 \lambda(\bar{Z}) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i \text{ or}$$

$$R[1 - x_c(R)] + \bar{Z}[x_c(R) - \bar{x}_i(R)] - \gamma[1 - \bar{x}_i(R)] = 0$$

PROPOSITION 2.

a. When  $\varepsilon \leq \varepsilon_c < 0$  with  $\varepsilon_c \equiv \frac{\bar{\theta}[(R - \gamma)(R - \bar{\theta})]}{\sigma[R(\gamma - \bar{Z}) + \bar{\theta}(R - \gamma)]}$  the bank goes bankrupt.

b. There is a value  $R_{\min}$  of the rate of return associated with a value of bank leverage  $\ell(R_{\min})$  that minimises the bank's probability of default.

Proof of Proposition 2: see Appendix.

The value  $\varepsilon_c$  can be considered as a measure of the financial fragility of the economy, as it defines the critical level of the macroeconomic shock for which the bank goes bankrupt. A rise in  $\varepsilon_c$  means that the bank is more sensitive to a shock in the sense that the value of the shock that is required to make it fail is lower: financial fragility increases. In our model, the degree of financial fragility depends on the overall business climate, namely the maximum level of return on financed projects ( $\bar{\theta}$ ), the risk-free interest rate ( $\gamma$ ) and the rate of return charged by the bank ( $R$ ).

It is important to note that there is a nonlinear relationship between financial fragility and the rate of return charged by the bank. In order to understand this result, let us first assume that the rate of return charged by the bank is high. This means that the value of leverage and the total quantity of financing are low. Let us now assume that the bank decides to cut the value of the rate of return it charges to firms. Two mechanisms take place. First, firms' *ex ante* probability of default falls with the rate of return charged by the bank. This first mechanism positively affects bank financial stability. Second, as the rate of return charged by the bank decreases, more firms are financed and the bank's regulatory capital increases with the quantity of financing. Nevertheless, as firms are financed partly by bank capital and partly by deposits, the value of leverage increases (see part b of proposition 1). As the leverage increases, the *ex post* value of the bank depends more and more on the value of its assets. This second mechanism negatively affects the bank's financial stability as the *ex post* value of the bank's assets are related

to the level of the macroeconomic shock. Consequently, there is a critical value of the rate of return charged by the bank beyond which the second effect outweighs the first effect and the bank becomes more sensitive to the value of the macroeconomic shock. This critical value of the rate of return charged by the bank is associated with a critical value of leverage for which financial fragility increases in line with the leverage.

$\ell(R_{\min})$  is thus defined as the “maximum stability value of leverage” which means the value of the equity multiplier for which the bank’s probability of default is at its minimum, and financial stability is at its maximum. Nevertheless, there is no reason for the bank to choose this specific value. On the contrary, we have shown that the equilibrium level of leverage chosen by the bank is the one that maximises its net RAROC,  $\ell(R^*)$ . Consequently, two situations are possible. In the first case,  $\ell(R^*) < \ell(R_{\min})$  and the equilibrium value of leverage chosen by the bank is lower than the “maximum stability value”. This situation is “inefficient” from the point of view of the economy as a whole, since it is possible to increase the quantity of financing and financial stability simultaneously. In fact, a lower rate of return charged by the bank will increase the quantity of funds available to firms. Simultaneously, this increase in the quantity of financing will lead to an increase in the level of bank leverage and a decrease in the bank’s probability of default. In the second case,  $\ell(R^*) > \ell(R_{\min})$  and the equilibrium value of leverage is higher than the “maximum stability value”. In that case, there is a trade-off between financial stability and credit availability, as a higher degree of financial fragility must be accepted in order to increase the quantity of credit available to firms above  $\ell(R_{\min})$ .<sup>8</sup>

In Figure 1, we give a graphical representation of this mechanism. The right quadrant describes the relationship between the rate of return charged by the bank and the value of bank leverage  $\ell(R)$ . This relationship is decreasing, as a decline in the rate of return charged by the bank leads to a rise in leverage. The left quadrant links the value of the bank’s leverage to its probability of default ( $\phi(\varepsilon_c)$ ), which is related to the critical value of the macroeconomic shock  $\varepsilon_c$ .

As the rate of return charged by the bank decreases, the quantity of funds available to firms increases and more projects can be undertaken. Simultaneously, this increase in the quantity of financing leads to a rise in the level of bank leverage and a decrease in the probability of default as long as  $\ell(R) < \ell(R_{\min})$ . When the level of bank leverage becomes higher than  $\ell(R_{\min})$ , the bank’s probability of default increases in line with

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<sup>8</sup> This result is in line with the one of Inderst and Mueller (2008) who show that leverage is beneficial, at least up to certain point, in order to give banks incentives to make new risky loans.

the level of financing. Consequently, from that point, higher credit availability is possible if one accepts a higher level of financial instability.

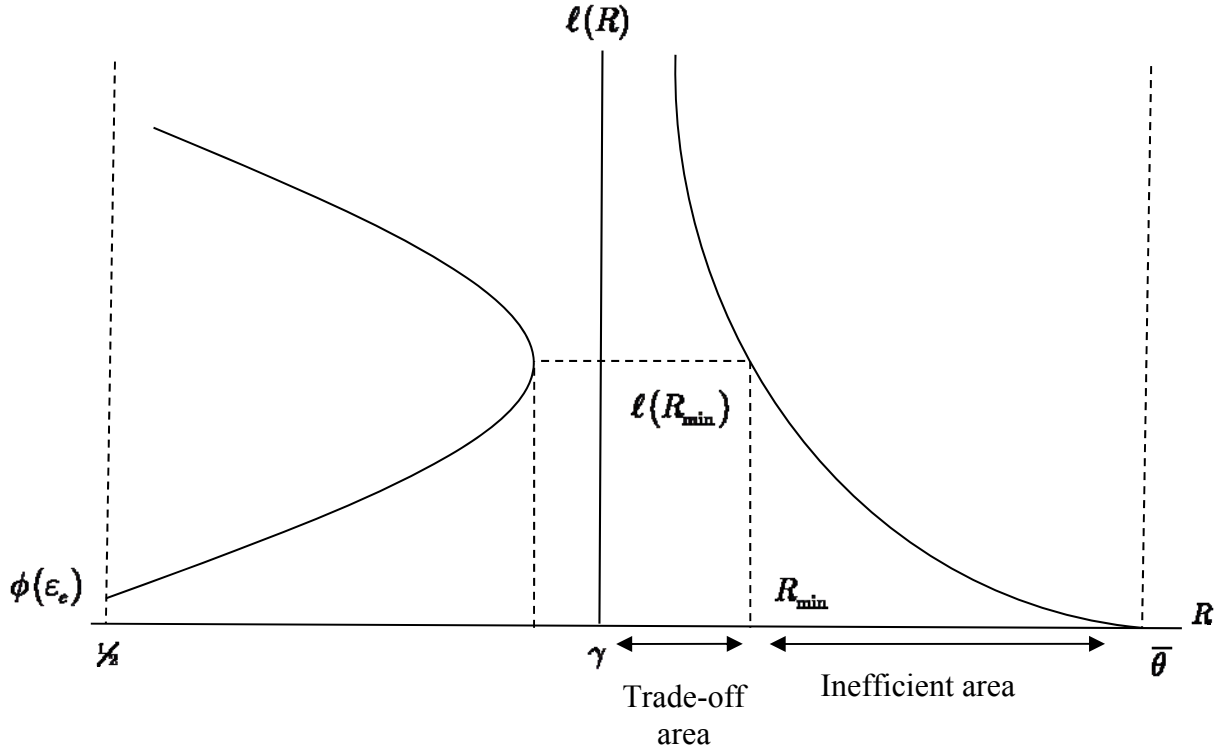


Fig. 1. Bank leverage and financial stability

#### 4. RISE IN ASSETS PRICES, FINANCIAL STABILITY AND MACROPRUDENTIAL REGULATION

It is clear that the equilibrium rate of return charged by the bank depends on the overall state of the economy. Below, we study the impact of a change in the expected value of the collateral on bank behaviour and financial stability. We show there is a critical value of the collateral for which financial fragility increases with a rise in assets prices. This means that the equilibrium value of leverage chosen by the bank becomes higher than "the maximum stability value" and continues to diverge with the bubble. We therefore estimate "the maximum stability value of leverage" for given parameter values. By this heuristic experiment we show that this "maximum stability value" is not fixed but depends on the overall business climate.

#### 4.1. Asset prices and financial stability

We model an rise in assets prices as an increase in the expected value of the collateral  $Z^e$ . It is possible to show that this increase has a positive impact on the total level of financing and the equilibrium leverage value chosen by the bank.

##### PROPOSITION 3

- a.  $\ell(R^*)$  is an increasing function of the expected value of the collateral  $Z^e$ .
- b. There is a critical value  $Z_c^e$  for which  $\ell(R^*) > \ell(R_{\min})$  and financial fragility increases along with the rise in the value of the collateral.

Proof of proposition 3: see Appendix.

Proposition 3 can be easily understood. The increase in the value of the collateral has a direct positive impact on the bank's net expected profit since, *ceteris paribus*, it increases the expected return in the event of a firm's default. Note also that a change in the expected value of the collateral directly alters the required level of regulatory capital (which decreases), since it depends on the LGD value estimated by the bank. Consequently, there is a kind of "freeing" amount of regulatory capital compared to the previous situation and the bank must change its behaviour in order to reach a new equilibrium. Actually, because of the "freeing" level of capital and the increase in the net expected profit for each loan that is financed, the bank is inclined to increase its level of financing. This can be done by cutting the rate of return charged on each loan. In this case, the quantity of funds provided to firms increases and the *ex ante* probability of default of each project falls as the rate of return charged to each firm decreases. At the same time, as the quantity of financing increases, the required level of regulatory capital increases. This process will stop as soon as the bank has restored the equilibrium value of its RAROC. Lastly, the equilibrium level of leverage increases in line with the quantity of financing (see proposition 1).

As the equilibrium value of leverage chosen by the bank increases with the rise in the value of collateral, there is a critical expected value of the collateral for which the bank's effective leverage becomes higher than the "maximum stability value of leverage" (part b of proposition 3). This means that financial fragility increases with the rise in assets prices as the bank becomes more and more sensitive to macroeconomic shocks. However, because of the structure of the model, it is impossible to formally obtain this critical expected value of the collateral. Thus, in the last part of the paper, we give a numerical illustration of proposition 3. This illustration is purely heuristic in the sense that we do not consider it as a prescriptive tool but as a way of stressing that both the "maximum stability value of leverage" and the critical expected value of the

collateral above which financial fragility increases depend on the overall business climate.

#### 4.2. A numerical illustration

We have already underlined that the Basel Committee has chosen a minimum leverage ratio of 3%, and thus a maximum equity multiplier of 33. These values seem to be consistent with historical average in non-crisis periods but they are not based on a specific economic reasoning (Jarrow, 2013). In this part of the paper, we provide a numerical illustration of proposition 3 in order to show that for some plausible values of the various parameters of the model, "maximum stability value of leverage" is far from 33.

Table 1 gives the parameters values adopted for the simulation, while Figure 2 gives a graphical representation of the "maximum stability value" and the effective levels of leverage retains by the bank.

| $\bar{\theta}$ | $\sigma$ | $\gamma$ | $\bar{Z}$ |
|----------------|----------|----------|-----------|
| 1.45           | 0.055    | 1.015    | 0.65      |

Table 1. Values of the parameters

We retain a debt recovery rate ( $\bar{Z}$ ) of 65% which is compatible with the mean recovery rate observed during financial crisis in Moody's or S&P's reports. We choose a riskless interest rate ( $\gamma$ ) of 1.5% which is near of the average rate of refinancing fixed by centrals banks between 2002 and 2005. Finally, the volatility of the macroeconomic shock ( $\sigma$ ) and the maximum rate of return on financed projects ( $\bar{\theta}$ ) are chosen in order to be compatible with a "good business climate" (5.5% and 45% respectively).

Figure 2 gives a numerical illustration of proposition 3 for the values of Table 1. The "maximum stability value of leverage" ( $\ell(R_{\min})$ ) is equal to 23.76 and corresponds to the horizontal line in Figure 2. It is far below the maximum equity multiplier of 33 fixed by Basel III prudential regulation (red line on Figure 2).

The increasing function represents the various equilibrium values of leverage chosen by the bank ( $\ell(R^*)$ ) according to the expected value of the collateral for the following range:  $Z^e \in [0.55; 0.75]$ .<sup>9</sup>

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<sup>9</sup> Simulations are done using Mathematica. Program is available on demand.



As expected, the equilibrium level of leverage that is chosen by the bank increases with the rise in the value of the collateral.

Figure 2 gives also a graphical illustration of the area of financial fragility of the economy. This area of financial fragility is defined as the equilibrium situations where the level of leverage chosen by the bank (the one that maximize its profits) is higher than the “maximum stability value of leverage”. This area is bounded by the critical expected value of the collateral which is equal, in this case, to 0.7.

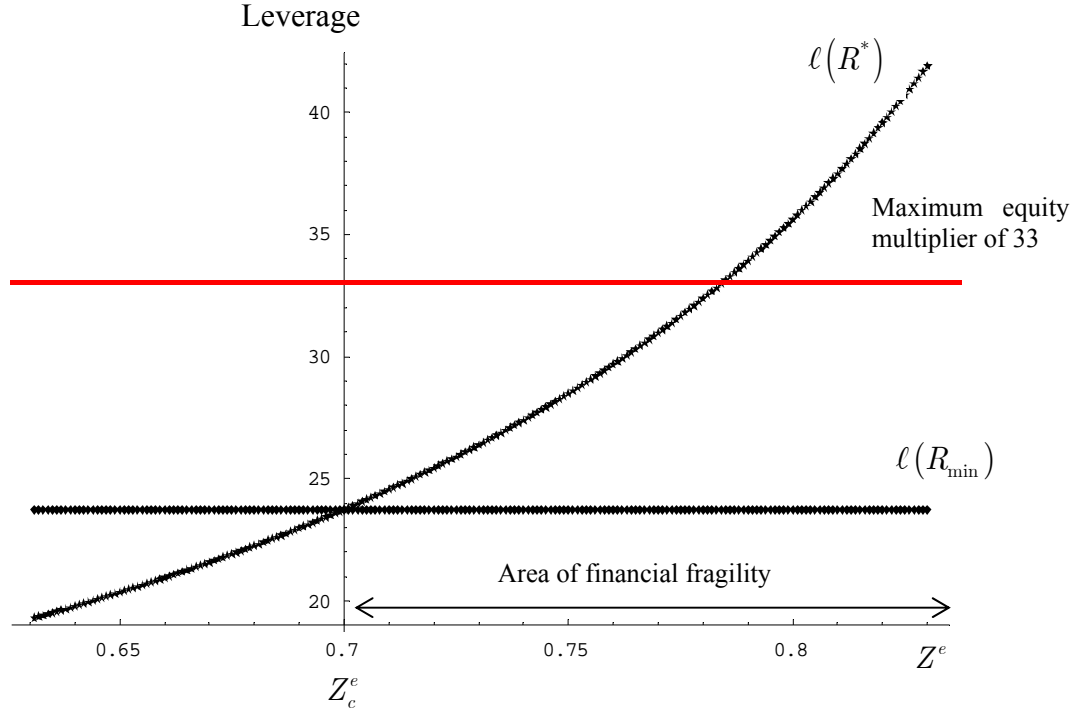


Fig. 2. Level of leverage and area of financial fragility

This result highlights that the choice of a fixed regulatory level for the leverage ratio may be misleading if the objective is to reinforce financial stability. Under specific macroeconomic conditions (business climate) a bank may choose a level of leverage lower than the one fixed by the new regulation but higher than the “maximum stability value of leverage”. In that case, financial stability is not guaranteed by the fulfilment of the new leverage ratio.

## 5. CONCLUDING COMMENTS

In this paper, we have shown that financial fragility may emerge even though it is assumed that banks make rational decisions under perfect information. Our results are twofold. First, we show that risk-sensitive microprudential regulation, such as Basel II, cannot prevent an increase in financial fragility due to bank behaviour. In periods of

expansion, characterised by a rise in assets prices, optimal bank behaviour leads to an increase in leverage that heightens financial fragility. Consequently, a maximum leverage ratio constraint seems justified in order to prevent financial fragility. This is the path followed by the new Basel III macroprudential regulations that aim to impose a maximum bank leverage of 33. However, our second result highlights the fact that the value of leverage that maximises financial stability is not constant along the cycle. This means that the regulator should adjust the leverage ratio in order to be efficient. This result is in line with the set of Basel III regulatory innovations which includes countercyclical capital ratios leading to stricter capital requirements during boom periods, in order to restrict the supply of loans (Arnold and Alii, 2012).

In Basel III, these countercyclical provisions must be calibrated not according to the specific exposure of each financial institution, but in response to the total exposure relating to the stage in the economic cycle. Rather than resorting to totally discretionary devices, as might be the case through an enriched Pillar 2 within Basel II, or, conversely, adopting automatic rules, the option chosen by the Basel Committee is to define guidelines, or targets (for instance the value of total loans on GDP), which if exceeded may justify a gradual increase in the capital requirements of Pillar 1. Since our model showed that the value of leverage that maximises financial stability is not constant within the cycle, we advocate for a similar approach based on targets to define an adjustable leverage ratio. Our results advocate for targets based on the level of the riskless interest rate, assets prices and macroeconomic volatility.

There are a number of directions in which our analysis can be extended but one stands out. Literature relative to risk-taking channel underlines that monetary policy affects risk-taking by banks because of the relationship between the level of short-term interest rates and the level of banking risks. This risk-taking channel seems to have played a major role in the run-up to the financial crisis. Empirically, even though the relationship is far from homogeneous, depending on the bank and the level of its capital, credit quality changes during the cycle and according to short-term rates. Thus, there is clearly a negative relationship between the interest rates set by monetary authorities and risk taking by banks, measured in different ways (spreads, banks' internal ratings). Low interest rates not only encourage the quantitative expansion of credit but also reduce its inherent quality in terms of risk.

These results are highlighted by the fact that, as we have stressed in this article, it seems impossible to determine the optimal value of the leverage ratio independently of the business climate and consequently of the short-term interest rates. This will require, at the very least, extensive coordination between central banks and supervisory authorities if they remain separate, wherever they are. Much still has to be done to redefine a new central banking system aiming to ensure both monetary stability and financial stability.

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## APPENDIX

### Preliminary

Remember that

$$K(R) = \int_{\bar{x}_i(R)}^1 k_i(R) dx_i = \int_{\bar{x}_i(R)}^1 \lambda(Z^e) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i$$

and we define

$$B(R) = \int_{\bar{x}_i(R)}^1 \left[ (1 - \phi(\bar{\varepsilon}_i(R))) R + \phi(\bar{\varepsilon}_i(R)) Z^e - \gamma \right] dx_i$$

With  $\bar{x}_i(R) = \frac{R}{\bar{\theta}}$ ,  $\bar{\varepsilon}_i(R) = \frac{R - \bar{\theta} x_i}{x_i \sigma_\theta}$  and  $P_i(\alpha, \bar{\varepsilon}_i) = \phi \left[ \frac{\bar{\varepsilon}_i + \sqrt{\rho} \varepsilon_\alpha}{\sqrt{(1 - \rho)}} \right]$ .

Note that  $K(R)$  and  $B(R)$  are continuous and differentiable on  $R \in [\gamma, \bar{\theta}]$ .

We know that for  $R \rightarrow F(R) = \int_{w(R)}^{u(R)} f(R, x) dx$  we have

$$F'_R(R) = \int_{w(R)}^{u(R)} \frac{\partial f}{\partial R}(R, x) dx + u'_R(R) \cdot f[R, u(R)] - w'_R(R) \cdot f[R, w(R)]$$

Consequently, we obtain

$$\frac{\partial K(R)}{\partial R} = K'_R(R) = \int_{\frac{R}{\bar{\theta}}}^1 \lambda(Z^e) \left( \left( \frac{\partial P_i(\alpha, \bar{\varepsilon}_i)}{\partial \bar{\varepsilon}_i} - \frac{\partial \phi(\bar{\varepsilon}_i)}{\partial \bar{\varepsilon}_i} \right) \frac{\partial \bar{\varepsilon}_i}{\partial R} \right) dx_i - \lambda(Z^e) \frac{1}{\bar{\theta}} [P_i(\alpha, 0) - \phi(0)]$$

and

$$K'_R(R) = \lambda(Z^e) \left[ \int_{\frac{R}{\bar{\theta}}}^1 \left[ (P_i(\alpha, \bar{\varepsilon}_i)'_{\varepsilon} - \phi'_{\varepsilon}(\bar{\varepsilon}_i)) \frac{1}{x_i \sigma_\theta} \right] dx_i - \frac{1}{\bar{\theta}} [P_i(\alpha, 0) - \phi(0)] \right]$$

$$B'_R(R) = \int_{\frac{R}{\bar{\theta}}}^1 \left[ (1 - \phi(\bar{\varepsilon}_i(R))) - R \frac{\partial \phi(\bar{\varepsilon}_i)}{\partial \bar{\varepsilon}_i} \frac{\partial \bar{\varepsilon}_i}{\partial R} + Z^e \frac{\partial \phi(\bar{\varepsilon}_i)}{\partial \bar{\varepsilon}_i} \frac{\partial \bar{\varepsilon}_i}{\partial R} \right] dx_i - \frac{1}{\bar{\theta}} [(1 - \phi(0)) R + \phi(0) Z^e - \gamma]$$

$$B'_R(R) = \int_{\frac{R}{\bar{\theta}}}^{\bar{x}} \left[ (1 - \phi(\bar{\varepsilon}_i(R))) - (R - Z^e) \frac{\phi'_{\varepsilon}(\bar{\varepsilon}_i)}{x_i \sigma_\theta} \right] dx_i - \frac{1}{\bar{\theta}} \left[ \frac{1}{2} (R + Z^e) - \gamma \right]$$

With  $\phi(0) = \frac{1}{2}$ ,  $\bar{x}'_i(R) = \frac{\partial \bar{x}_i(R)}{\partial R} = \frac{1}{\bar{\theta}} > 0$  since  $\bar{x}_i(R) = \frac{R}{\bar{\theta}}$ ,  $\frac{\partial \phi(\bar{\varepsilon}_i)}{\partial \bar{\varepsilon}_i} = \phi'_\varepsilon(\bar{\varepsilon}_i) > 0$ ,  
 $\frac{\partial \bar{\varepsilon}_i}{\partial R} = \frac{1}{x_i \sigma_\theta}$  as  $\bar{\varepsilon}_i(R) = \frac{R - \bar{\theta} x_i}{x_i \sigma_\theta}$  and  $\frac{\partial P_i(\alpha, \bar{\varepsilon}_i)}{\partial \bar{\varepsilon}_i} = P'_i(\alpha, \bar{\varepsilon}_i) > 0$ .

**Lemma 1.**

For  $2\gamma - Z^e < \bar{\theta} < \frac{5\gamma\sigma_\theta}{(\gamma - Z^e)}$ , there is  $\bar{R}$  such that

a. For  $R \in [\gamma, \bar{R}]$  we have  $B'_R(R) > 0$

b. For  $R \in ]\bar{R}, \bar{\theta}]$  we have  $B'_R(R) < 0$

and  $\bar{R}$  is the unique value of the interest rate for which  $B(R)$  is maximum.

**Proof of Lemma 1.**

$$B'_R(\bar{\theta}) = -\frac{1}{\bar{\theta}} \left[ \frac{1}{2}(\bar{\theta} + Z^e) - \gamma \right] < 0 \text{ since}$$

$$\int_1^{\frac{1}{\bar{\theta}}} \left[ (1 - \phi(\bar{\varepsilon}_i(R))) - (R - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i)}{x_i \sigma_\theta} \right] dx_i = 0 \quad \text{and} \quad \frac{1}{2}(\bar{\theta} + Z^e) - \gamma > 0 \quad \text{for}$$

$$\bar{\theta} > 2\gamma - Z^e.$$

$$B'_R(\gamma) = \int_{\frac{\gamma}{\bar{\theta}}}^1 \left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i \sigma_\theta} \right] dx_i - \frac{1}{\bar{\theta}} \left[ -\frac{1}{2}(\gamma - Z^e) \right]$$

Note that  $-\frac{1}{\bar{\theta}} \left[ -\frac{1}{2}(\gamma - Z^e) \right] > 0$  as  $\gamma > Z^e$ .

Thus, it is sufficient to prove that  $\int_{\frac{\gamma}{\bar{\theta}}}^1 \left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i \sigma_\theta} \right] dx_i > 0$  in

order for  $B'_R(\gamma) > 0$ .

Let us prove that  $\left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i \sigma_\theta} \right] > 0, \forall x_i \in \left[ \frac{\gamma}{\bar{\theta}}, 1 \right]$ .

We know that  $(1 - \phi(\bar{\varepsilon}_i(\gamma)))$  reaches its minimum, equal to  $(1 - \phi(0)) = \frac{1}{2}$ , for

$x_i = \frac{\gamma}{\theta}$  as  $\bar{\varepsilon}_i(\gamma) = \frac{R - \bar{\theta}x_i}{x_i\sigma_\theta} = 0$ . Moreover,  $(1 - \phi(\bar{\varepsilon}_i(\gamma)))$  is increasing with  $x_i$ .

Recall that  $\phi'_\varepsilon(\bar{\varepsilon}_i(R)) = \frac{\partial \phi(\bar{\varepsilon}_i(R))}{\partial \bar{\varepsilon}_i(R)}$  with  $\phi(\bar{\varepsilon}_i(R)) = \int_{-\infty}^{\bar{\varepsilon}_i(R)} \varphi(t) dt$  and  $\varphi(t)$  the density function of a normally distributed stochastic variable. Thus we have  $\phi'_\varepsilon(\bar{\varepsilon}_i(R)) = \varphi(\bar{\varepsilon}_i(R))$  and  $(\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta} = (\gamma - Z^e) \frac{\varphi(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta}$  reaches its maximum for  $x_i = \frac{\gamma}{\theta}$  as  $\phi'_\varepsilon(0) = \varphi(0) = 0,4$  and is decreasing when  $x_i$  increases.

Consequently, if  $\left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta} \right] > 0$  for  $x_i = \frac{\gamma}{\theta}$ , we are sure

that  $\left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta} \right] > 0, \forall x_i \in \left[ \frac{\gamma}{\theta}, 1 \right]$  and

$$\int_{\frac{\gamma}{\theta}}^1 \left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta} \right] dx_i > 0.$$

For  $x_i = \frac{\gamma}{\theta}$  we have

$$\left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta} \right] = (1 - \phi(0)) - (\gamma - Z^e) \bar{\theta} \frac{\varphi(0)}{\gamma\sigma_\theta} > 0 \text{ if } (\gamma - Z^e) \bar{\theta} \frac{0,4}{\gamma\sigma_\theta} < \frac{1}{2} \text{ which is true for } \bar{\theta} < \frac{5\gamma\sigma_\theta}{(\gamma - Z^e)}.$$

Consequently, we have

$$B'_R(\gamma) = \int_{\frac{\gamma}{\theta}}^1 \left[ (1 - \phi(\bar{\varepsilon}_i(\gamma))) - (\gamma - Z^e) \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(\gamma))}{x_i\sigma_\theta} \right] dx_i - \frac{1}{\theta} \left[ -\frac{1}{2}(\gamma - Z^e) \right] > 0$$

Finally, we have to prove that  $B''(R) < 0, \forall R$ .

$$B''(R) = - \int_{\frac{R}{\bar{\theta}}}^1 \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(R))}{x_i\sigma_\theta} dx_i - \frac{1}{2\bar{\theta}} - \int_{\frac{R}{\bar{\theta}}}^1 \left[ \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(R))}{x_i\sigma_\theta} + (R - Z) \frac{\phi''_\varepsilon(\bar{\varepsilon}_i(R))}{(x_i\sigma_\theta)^2} \right] dx_i - \frac{1}{2\bar{\theta}}$$

which leads, after simplification, to

$$B''(R) = -2 \int_{\frac{R}{\bar{\theta}}}^1 \frac{\phi'_\varepsilon(\bar{\varepsilon}_i(R))}{x_i \sigma_\theta} dx_i - \frac{1}{\bar{\theta}} - \int_{\frac{R}{\bar{\theta}}}^1 (R - Z^e) \frac{\phi''_\varepsilon(\bar{\varepsilon}_i(R))}{(x_i \sigma_\theta)^2} dx_i < 0 \quad \text{since}$$

$\phi''_\varepsilon(\bar{\varepsilon}_i(R)) \geq 0$  for  $\bar{\varepsilon}_i(R) \leq 0$  which is always the case.

Thus, as  $B''(R) < 0 \quad \forall R$ ,  $B(R)$  is concave on  $R \in [\gamma, \bar{\theta}]$  and  $\bar{R}$  is the unique maximum of  $B(R)$ .

Moreover, as  $B(\gamma) < 0$ ,  $B(\bar{R}) > 0$  and  $B(\bar{\theta}) = 0$ , there is  $R_c \in ]\gamma, \bar{R}[$  such that  $B(R_c) = 0$  and  $B(R) \geq 0 \quad \forall R \in [R_c, \bar{\theta}]$ .

The proof of Lemma 1 is completed ■

## Lemma 2.

For  $R \in [\gamma, \bar{\theta}]$  we have  $K'_R(R) < 0$ .

### Proof of Lemma 2.

$$K'_R(R) = \lambda(Z^e) \left[ \int_{\frac{R}{\bar{\theta}}}^1 \left[ \left( P_i(\alpha, \bar{\varepsilon}_i)'_\varepsilon - \phi'_\varepsilon(\bar{\varepsilon}_i) \right) \frac{1}{x_i \sigma_\theta} \right] dx_i - \frac{1}{\bar{\theta}} [P_i(\alpha, 0) - \phi(0)] \right]$$

We know that  $\phi'_\varepsilon(\bar{\varepsilon}_i) > 0$  and  $P_i(\alpha, \bar{\varepsilon}_i)'_\varepsilon > 0$ . Moreover, we have

$$P_i(\alpha, \bar{\varepsilon}_i)'_\varepsilon < \phi'_\varepsilon(\bar{\varepsilon}_i), \quad \forall \bar{\varepsilon}_i \quad \text{as} \quad P_i(\alpha, \bar{\varepsilon}_i) = \phi \left[ \frac{\bar{\varepsilon}_i + \sqrt{\rho} \varepsilon_\alpha}{\sqrt{(1-\rho)}} \right] \quad \text{and} \quad \frac{\partial \rho}{\partial \bar{\varepsilon}_i} < 0 \quad \text{since by}$$

construction of the IRB approach, the sensitivity to systematic risk decreases when the specific risk of the project increases. Consequently, we have

$$\left[ \left( P_i(\alpha, \bar{\varepsilon}_i)'_\varepsilon - \phi'_\varepsilon(\bar{\varepsilon}_i) \right) \frac{1}{x_i \sigma_\theta} \right] < 0, \quad \forall x_i. \quad \text{Finally, as } \frac{1}{\bar{\theta}} [P_i(\alpha, 0) - \phi(0)] > 0 \text{ we have}$$

$$K'_R(R) = \lambda(Z^e) \left[ \int_{\frac{R}{\bar{\theta}}}^1 \left[ \left( P_i(\alpha, \bar{\varepsilon}_i)'_\varepsilon - \phi'_\varepsilon(\bar{\varepsilon}_i) \right) \frac{1}{x_i \sigma_\theta} \right] dx_i - \frac{1}{\bar{\theta}} [P_i(\alpha, 0) - \phi(0)] \right] < 0$$

The proof of Lemma 2 is completed ■



**Proof of part a. of Proposition 1.**

In order to prove the existence of a unique maximum for  $\Gamma(R) = \frac{B(R)}{K(R)}$ , we use Darboux's Theorem.

$\Gamma(R)$  is continuous and differentiable on  $R \in [R_c, \bar{\theta}]$ .

According to Lemma 1 and 2 we have:

$$\Gamma'(R_c) = \frac{\partial \Gamma(R)}{\partial R}(R_c) = \frac{B'(R_c)K(R_c) - B(R_c)K'(R_c)}{K^2(R_c)} = \frac{B'(R_c)}{K(R_c)} > 0 \text{ as } B(R_c) = 0$$

and  $B'(R_c), K(R_c) > 0$

$$\lim_{R \rightarrow \bar{\theta}} \Gamma'(R) = \frac{\partial \Gamma(R)}{\partial R}(\bar{\theta}) = \frac{B'(\bar{\theta})K(\bar{\theta}) - B(\bar{\theta})K'(\bar{\theta})}{K^2(\bar{\theta})} = \frac{B'(\bar{\theta})}{K(\bar{\theta})} \rightarrow -\infty \text{ as } B(\bar{\theta}) = 0,$$

$K(\bar{\theta}) = 0$ , and  $B'(\bar{\theta}) < 0$ .

The conditions of Darboux's Theorem are fulfilled and we can conclude that there is a unique  $R^* \in ]R_c, \bar{\theta}[$  such that  $\Gamma(R^*) = 0$  and  $R^*$  is a maximum.

Moreover, since  $R^* > R_c$  we have  $B(R^*) > 0$ ,  $K(R^*) > 0$  and  $\Gamma(R^*) > 0$

**Part b of proposition 1** is obvious as  $\frac{\partial \bar{x}_i(R^*)}{\partial R^*} = \frac{1}{\bar{\theta}} > 0$  implying that

$$\frac{\partial D(R^*)}{\partial R^*} = D'(R^*) < 0.$$

Concerning part c of proposition 1,  $\frac{\partial \ell^*}{\partial R^*} = \frac{D'(R^*)K(R^*) - D(R^*)K'(R^*)}{K^2(R^*)} < 0$  since

$$\frac{D'(R^*)}{D(R^*)} > \frac{K'(R^*)}{K(R^*)} \text{ as bank capital finances only a part of the new projects.}$$

Proof of proposition 1 is completed ■

## Proof of proposition 2.

### Proof of part a.

The ex-post value of the bank is given by

$$V_b(R) = \int_{\bar{x}_i(R)}^1 \lambda(\bar{Z}) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i + R[1 - x_c(R)] + \bar{Z}[x_c(R) - \bar{x}_i(R)] - \gamma[1 - \bar{x}_i(R)]$$

and the bank goes bankrupt when

$$V_b(R) = \int_{\bar{x}_i(R)}^1 \lambda(\bar{Z}) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i \text{ or}$$

$$R[1 - x_c(R)] + \bar{Z}[x_c(R) - \bar{x}_i(R)] - \gamma[1 - \bar{x}_i(R)] = 0$$

with  $x_c(R) = \frac{R}{\bar{\theta} + \varepsilon_c \sigma_\theta}$ ,  $\bar{x}_i(R) = \frac{R}{\bar{\theta}}$ . Substituting these values in the first equation we

obtain

$$R \left[ 1 - \frac{R}{\bar{\theta} + \varepsilon_c \sigma_\theta} \right] + \bar{Z} \left[ \frac{R}{\bar{\theta} + \varepsilon_c \sigma_\theta} - \frac{R}{\bar{\theta}} \right] - \gamma \left[ 1 - \frac{R}{\bar{\theta}} \right] = 0 \text{ for}$$

$$\varepsilon_c = \frac{\bar{\theta} [(R - \gamma)(R - \bar{\theta})]}{\sigma [R(\gamma - \bar{Z}) + \bar{\theta}(R - \gamma)]}$$

We know that  $\bar{\theta} > 0$ ,  $(R - \gamma) > 0$ ,  $(R - \bar{\theta}) < 0$ , and  $\bar{\theta} [(R - \gamma)(R - \bar{\theta})] < 0$ .

Moreover, as  $\gamma > \bar{Z}$  and  $R > \gamma$  we have  $\sigma [R(\gamma - \bar{Z}) + \bar{\theta}(R - \gamma)] > 0$  and  $\varepsilon_c < 0$  ■

### Proof of part b.

We search for the value of  $R$  that minimises the bank's probability of default. We know that this probability of default decreases with the value of  $\varepsilon_c$ .

$$\frac{\partial \varepsilon_c}{\partial R} = \frac{\bar{\theta} [\gamma(\bar{Z} - 2R\bar{\theta}) + R^2(\bar{\theta} + \gamma - \bar{Z})]}{\sigma [R(\gamma - \bar{Z}) + \bar{\theta}(R - \gamma)]^2} = 0 \text{ for}$$

$$R_1 = \frac{\bar{\theta}\gamma - \sqrt{\gamma\bar{\theta}(\gamma - \bar{Z})(\theta - \bar{Z})}}{\gamma + \bar{\theta} - \bar{Z}}$$

$$R_2 = \frac{\bar{\theta}\gamma + \sqrt{\gamma\bar{\theta}(\gamma - \bar{Z})(\theta - \bar{Z})}}{\gamma + \bar{\theta} - \bar{Z}}$$

With  $\gamma\bar{\theta}(\gamma - \bar{Z})(\theta - \bar{Z}) > 0$ . Moreover, it is obvious that  $R_2 > 1$ ,  $R_1 < 1$  and

$$\frac{\partial^2 \varepsilon_c}{\partial R^2} = \frac{-2\bar{\theta}^2\gamma[(\bar{Z} - \bar{\theta})(\gamma - \bar{Z})]}{\sigma[R(\gamma - \bar{Z}) + \bar{\theta}(R - \gamma)]^3} > 0.$$

Consequently, there is a unique value,  $R_{\min} = \frac{\bar{\theta}\gamma + \sqrt{\gamma\bar{\theta}(\gamma - \bar{Z})(\theta - \bar{Z})}}{\gamma + \bar{\theta} - \bar{Z}} > 1$  that minimises the value of  $\varepsilon_c$ .

Finally, there is a unique value of bank leverage  $\ell(R_{\min}) = \frac{D(R_{\min})}{K(R_{\min})}$  that minimises the

bank's probability of default and the relation between the bank's leverage and the bank's probability of default is nonlinear ■

### Proof of proposition 3.

#### Part a.

$$\ell(R^*) = \frac{D(R^*, Z^e)}{K(R^*, Z^e)} \text{ and } \frac{\partial \ell(R^*)}{\partial Z^e} = \frac{\frac{\partial D(\bullet)}{\partial Z^e} K(\bullet) - D(\bullet) \frac{\partial K(\bullet)}{\partial Z^e}}{K(\bullet)^2}.$$

As  $K(R^*, Z^e) > 0$ ,  $D(R^*, Z^e) > 0$  and  $\frac{\partial K(R^*, Z^e)}{\partial Z^e} < 0$  we are sure that  $\frac{\partial \ell(R^*)}{\partial Z^e} > 0$

if  $\frac{\partial D(R^*, Z^e)}{\partial Z^e} > 0$ . As  $D(R^*, Z^e) = \left(1 - \frac{R^*}{\bar{\theta}}\right)$ ,  $\frac{\partial D(R^*, Z^e)}{\partial Z^e} > 0$  if  $\frac{\partial R^*}{\partial Z^e} < 0$ .

We have  $\Gamma(R^*, Z^e) = \frac{B(R^*, Z^e)}{K(R^*, Z^e)}$  the maximum value of the RAROC for a given

expected value of the collateral. Takes the total derivative of  $\Gamma(R^*, Z^e)$  and equates it to zero

$$d\Gamma(R^*, Z^e) = \frac{\partial \Gamma(R^*, Z^e)}{\partial Z^e} dZ^e + \frac{\partial \Gamma(R^*, Z^e)}{\partial R^*} dR^* = 0 \text{ and}$$

$$\frac{dR^*}{dZ^e} = - \frac{\frac{\partial \Gamma(R^*, Z^e)}{\partial Z^e}}{\frac{\partial \Gamma(R^*, Z^e)}{\partial R^*}}$$

$$\frac{\partial \Gamma(R^*, Z^e)}{\partial R^*} = 0$$

$$\frac{\partial \Gamma(R^*, Z^e)}{\partial Z^e} = \frac{\frac{\partial B(\bullet)}{\partial Z^e} K(\bullet) - B(\bullet) \frac{\partial K(\bullet)}{\partial Z^e}}{K(\bullet)^2} > 0 \text{ as}$$

$$K(R^*, Z^e) > 0, B(R^*, Z^e) > 0, \frac{\partial K(R^*, Z^e)}{\partial Z^e} < 0 \text{ and}$$

$$\frac{\partial B(\bullet)}{\partial Z^e} = \int_{\bar{x}_i(R^*)}^1 \phi(\bar{\varepsilon}_i(R^*)) dx_i > 0.$$

$$\text{And } \frac{dR^*}{dZ^e} < 0.$$

### Part b.

According to proposition 1, there is a equilibrium rate of return for the bank if  $\bar{\theta} > 2\gamma - Z^e$  or, put differently, for  $Z^e > 2\gamma - \bar{\theta} < 1$  with  $\bar{\theta} > 2\gamma - 1$ .

Thus, for  $Z^e < 2\gamma - \bar{\theta} < 1$  there is no equilibrium and  $\ell(R^*) = 0$ .

Moreover, as  $K(R) = \int_{\bar{x}_i(R)}^1 \lambda(Z^e) (P_i(\alpha, \bar{\varepsilon}_i) - \phi(\bar{\varepsilon}_i)) dx_i$  and  $\lim_{Z^e \rightarrow 1} \lambda(Z^e) = 0$ , we have

$$\lim_{Z^e \rightarrow 1} \ell(R^*) \rightarrow +\infty.$$

Consequently, as  $\ell(R_{\min}) = \frac{D(R_{\min}, \bar{Z})}{K(R_{\min}, \bar{Z})}$  with  $R_{\min} = \frac{\bar{\theta}\gamma + \sqrt{\gamma\bar{\theta}(\gamma - \bar{Z})(\theta - \bar{Z})}}{\gamma + \bar{\theta} - \bar{Z}}$  we

have  $0 < \ell(R_{\min}) < \infty$  and we are sure there is  $Z_c^e \in [2\gamma - \bar{\theta}; 1]$  such that  $\ell(R^*(Z_c^e)) = \ell(R_{\min})$  and  $\ell(R^*(Z^e)) > \ell(R_{\min})$  for  $Z^e > Z_c^e$ .

The proof of proposition 3 is completed ■